Stable sets, corner polyhedra and the Chvátal closure*

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Abstract

In this work, we consider a classical formulation of the stable set problem. We characterize its corner polyhedron, i.e. the convex hull of the points satisfying all the constraints except the non-negativity of the basic variables. We show that the non-trivial inequalities necessary to describe this polyhedron can be derived from one row of the simplex tableau as fractional Gomory cuts. It follows in particular that the split closure is not stronger than the Chvátal closure for the stable set problem. The results are obtained via a characterization of the basis and its inverse in terms of a collection of connected components with at most one cycle.

1 Introduction

Consider a simple graph $G = (V, E)$, where V and E are the sets of n vertices and m edges of G, respectively. A *stable set* (independent set, vertex packing) of G is a set of pairwise non-adjacent vertices. For the sake of simplicity, we are going to assume that G has no connected component defined by a single vertex. Then, a stable set corresponds to an n-dimensional binary vector x that satisfies $x_u + x_v \leq 1$, for all $uv \in E$. The set of stable sets of G can be described by the mixed integer linear set

$$S(G) = \{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^m_+ : \ Ax + y = \mathbf{1}\}, \tag{1}$$

where A is the edge-vertex incidence matrix of G, $\mathbf{1}$ is a vector of ones, x and y are vectors of variables indexed by the vertices and the edges of G, respectively. Notice that

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the integrality of the slack variables y is enforced by the integrality of x. The convex hull of $S(G)$ is called the stable set polytope. The set obtained from $S(G)$ by relaxing the integrality constraints will be denoted by $R(G)$.

An important reason for studying methods to handle contraints of type (1) is that they can model restrictions appearing in many optimization problems. Indeed, notice that set packing and partitioning problems can be transformed into vertex-packing problems on the intersection graph.

A general approach to deal with mixed integer sets consists of solving the linear relaxation and adding cutting planes. Most commonly, these cutting planes are derived from integrality arguments applied to a single equation. Intersection cuts are more general [4]. Recently, cutting planes derived from two or more rows have attracked renewed interest, and it has been shown that they can better approximate the feasible set [2, 3, 5, 10]. Whether they are derived from one or more rows by integrality arguments, the known cuts have in common the feature that they are valid for the corner polyhedron defined by Gomory [12]. In this paper, we study the corner polyhedron associated to (1) and, consequently, investigate cuts that can be derived from a basic solution of the linear relaxation of (1).

In the remainder, let $\mathcal{B}$ stand for the set of all bases of the constraint matrix $[A \quad I]$. We denote by B an element of $\mathcal{B}$ and by N the resulting nonbasic submatrix. We represent the basic and nonbasic variables by $z_B = \begin{bmatrix} x_B \\ y_B \end{bmatrix}$ and $z_N = \begin{bmatrix} x_N \\ y_N \end{bmatrix}$, respectively. Without loss of generality, we are assuming that, in B and N, the columns indexed by vertices appear before those indexed by edges. Using this notation, constraints (1) can be rewritten equivalently as:

$$z_B = B^{-1}\mathbf{1} - B^{-1}N z_N, \quad z_B \geq 0, z_N \geq 0, x \in \mathbb{Z}^n, \tag{2}$$

Discarding the non-negativity constraints $z_B \geq 0$ of the basic variables, we get a relaxation of (2). The convex hull of the resulting set of points is the so-called *corner polyhedron* associated with basis B, to be denoted by corner(B). If the basic solution obtained by setting $z_N = 0$ is not integral, then it does not belong to this polyhedron, and a valid inequality for corner(B) cuts off this basic solution.

In this paper, we show that all valid inequalities necessary to the description of corner(B) can be derived from one row of (2) as a Chvátal-Gomory cut [9, 11]. In addition, we relate the intersection of the corner polyhedra associated to all bases, given by

$$\text{corner}(\mathcal{B}) = \bigcap_{B \in \mathcal{B}} \text{corner}(B),$$

to the split, Chvátal and $\{0, 1/2\}$-Chvátal closures relative to $S(G)$. The Chvátal closure is the intersection of $R(G)$ with all the Chvátal inequalities, i.e. inequalities of the form $\lfloor \lambda A \rfloor x \leq \lfloor \lambda \mathbf{1} \rfloor$ with $\lambda \in \mathbb{R}^m_+$ [9]. When λ is restricted to be in $\{0, 1/2\}^m$, the $\{0, 1/2\}$-Chvátal inequalities and closure are similarly defined [8].

It follows from [1] that the polyhedron corner$(\mathcal{B})$ is included in the split closure. In turn, the split closure is included in the Chvátal closure, which is itself contained in the $\{0, 1/2\}$-Chvátal closure. Here, we show the converse inclusions and obtain the following theorem.

Theorem 1. *For the stable set formulation* (1)*, the* $\{0, 1/2\}$*-Chvátal closure, the Chvátal closure, the split closure and corner*$(\mathcal{B})$ *are all identical to*

$$\{(x, y) \geq 0 : Ax + y = \mathbf{1}, \sum_{e \in \mathcal{C}} y_e \geq 1, \forall \text{induced odd cycle } C \text{ of } G\}.$$

In order to obtain these properties, we first describe the structure of the basic matrix and its inverse in the following two sections. Then, Section 4 is devoted to the characterization of the corner polyhedra, whereas Section 5 relates them to the split, Chvátal and $\{0, 1/2\}$-Chvátal closures.

2 The structure of the basis

Consider any basis $B \in \mathcal{B}$, feasible or infeasible. Let V_B and V_N represent the sets of basic and nonbasic vertices, that is, the vertices indexing columns of B and N, respectively. Similarly, define E_B and E_N with respect to the edges. In this and the next section, we assume that $V_B \neq \emptyset$, otherwise $B = I$ and we are done.

Let us consider the equations in (1) ordered in such a way that those indexed by E_B come last. Then, the basic matrix B has the form

$$B = \begin{bmatrix} \bar{B} & 0 \\ D & I \end{bmatrix}, \tag{3}$$

where the first group of columns is indexed by V_B and the second one is indexed by E_B.

In order to characterize $\bar{B}$, let $C_1, C_2, \ldots, C_k$ be the connected components of $G[V_B] \setminus E_B$, where $G[V_B]$ stands for the subgraph of G induced by V_B. For each $i = 1, 2, \ldots, k$, assume that $C_i = (V_i, E_i)$ and define E'_i as the set of nonbasic edges with (at least) one endpoint in V_i . Notice that $E'_i \supseteq E_i$ and $\{E'_i\}_{i=1}^k$ is a partition of E_N. This last part comes from the definition of the C_i's and the fact that B has no zero row.

Using this decomposition of the graph, we can partition submatrix $\bar{B}$ in a block-diagonal form as:

$$\bar{B} = \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{bmatrix}, \tag{4}$$

where each matrix B_i has its columns and rows respectively indexed by V_i and E'_i.

Lemma 1. *For every* $i = 1, 2, \ldots, k$*, the matrix* B_i *is square, that is,* $|V_i| = |E'_i|$*.*

Proof: Suppose that $|V_i| \neq |E'_i|$, for some i. Since $\bar{B}$ is a square matrix, we can assume that $|V_i| > |E'_i|$. Then, the columns of B_i must be linearly dependent, and so are the corresponding columns in $\bar{B}$. Therefore, $\det B = \det \bar{B} = 0$, which contradicts the fact that B is invertible. □

Now, we show that submatrices B_i are edge-vertex incidence matrices of pseudotrees. A *pseudotree* is a connected graph with at most one cycle. If it contains exactly one cycle, it is called a 1-*tree*

Lemma 2. *For every $i = 1, 2, \ldots, k$, either*

1. *C_i is a 1-tree with an odd cycle, and B_i is its edge-vertex incidence matrix; or*
2. *C_i is a tree, and B_i is the edge-vertex incidence matrix of C_i together with a foreign edge, that is, an edge linking a vertex in V_i to a vertex in V_N.*

Proof: Since C_i is connected, $|E_i| \geq |V_i| - 1$. By Lemma 1, it follows that $|E_i'| \geq |E_i| \geq |E_i'| - 1$. If $|V_i| = |E_i'| = |E_i|$, then the difference between C_i and a spanning tree of C_i is an edge. Moreover, the unique existing cycle must be odd because $\det B_i \neq 0$ [14]. This shows the first part of the statement. In the complementary case, $|V_i| = |E_i'| = |E_i| + 1$ and thus C_i is a tree. In this case, there is a row of B_i with exactly one entry equal to 1. Such a row corresponds to an edge of G with one endpoint in V_i and the other endpoint in V_N. This gives the second alternative of the lemma. $\square$

From now on, let I_0 and I_1 be complementary subsets of $\{1, 2, \ldots, k\}$ which index the tree and the 1-tree components, respectively. We define the kernel of C_i, denoted by $\kappa(C_i)$, as its unique cycle (if $i \in I_1$) or its foreign edge (if $i \in I_0$). The distance from a vertex v of C_i to its kernel is the length of the shortest path in C_i between v and a vertex in $\kappa(C_i)$. This vertex will be denoted κ_v. We simply write $u \in \kappa(C_i)$ or $e \in \kappa(C_i)$ to mean that a vertex u or an edge e is in the graph $\kappa(C_i)$.

Lemma 3. *Let $i \in I_0 \cup I_1$. The matrix B_i has the form*

$$B_i = \begin{bmatrix} H & & & & & \\ X_1 & I_{n_1} & & & & \\ & X_2 & I_{n_2} & & & \\ & & \ddots & \ddots & & \\ & & & & I_{n_{p-1}} & \\ & & & & X_p & I_{n_p} \end{bmatrix}, \tag{5}$$

where H is either equal to $[1]$ (if $i \in I_0$) or is the edge-vertex incidence matrix of the odd cycle in C_i (if $i \in I_1$) and, for $d = 1, 2, \ldots, p$, n_d is the number of vertices at distance d from $\kappa(C_i)$ and X_d is a matrix that has exactly one non-zero entry (equal to 1) per row. A non-zero entry of X_d is indexed by (uv, v), where the vertex v is the endpoint of the edge uv closest to $\kappa(C_i)$.

Proof: Let p be the maximum distance from a vertex of C_i to its kernel. For $d = 0, 1, \ldots, p$, let L_d be the set of vertices of C_i at distance d from $\kappa(C_i)$. L_p comprises leaves of C_i and so indexes columns of B_i which are different columns of the identity matrix. This gives the last column in (5). Similarly, L_{p-1} is formed by leaves of $C_i \setminus L_p$. Also, it contains all the vertices that are adjacent to some vertex in L_p. Such properties lead to the form of X_p. Continuing this process from the subgraph $C_i \setminus (L_p \cup L_{p-1})$, we get the remaining submatrix. At the final step, the vertices in L_0 are exactly those of the kernel. They define the submatrix H. If C_i is a tree with a foreign edge, the kernel is a vertex and so $H = [1]$. Otherwise, the kernel of C_i is the odd cycle of C_i. $\square$

3 The inverse of the basis

Lemma 1 and (3)–(4) establish that the basis inverse has the form

$$B^{-1} = \begin{bmatrix} \bar{B}^{-1} & 0 \\ -D\bar{B}^{-1} & I \end{bmatrix} \tag{6}$$

with

$$\bar{B}^{-1} = \begin{bmatrix} B_1^{-1} & & & \\ & B_2^{-1} & & \\ & & \ddots & \\ & & & B_k^{-1} \end{bmatrix}. \tag{7}$$

By partitioning D accordingly to $\bar{B}^{-1}$, it is easy to see that a non-zero row of any block of the matrix $D\bar{B}^{-1}$ is either a row of B_i^{-1} or the sum of two rows of B_i^{-1}, for some i. This is because each row of D has at most two non-zero entries, which are actually equal to 1. Thus, the inverse of the basis is essentially defined by the inverses B_i^{-1}, for $i = 1, 2, \ldots, k$. Lemma 3 implies that these matrices are given by

$$B_i^{-1} = \begin{bmatrix} H^{-1} & & & & & \\ J_{11} & I_{n_1} & & & & \\ J_{21} & J_{22} & I_{n_2} & & & \\ J_{31} & J_{32} & J_{33} & & & \\ & & & \ddots & \ddots & \\ J_{p1} & J_{p2} & J_{p3} & \cdots & J_{pp} & I_{n_p} \end{bmatrix}, \tag{8}$$

where, for $j = 1, \ldots, p$ and $l = j, \ldots, p$,

$$J_{lj} = \begin{cases} (-1)^{l-j+1} X_l X_{l-1} \cdots X_{j-1} X_j H^{-1}, & \text{if } j = 1 \\ (-1)^{l-j+1} X_l X_{l-1} \cdots X_{j-1} X_j I_{n_{j-1}}, & \text{if } j > 1 \end{cases} \tag{9}$$

We relate below the entries of B_i^{-1} to the graph C_i. The rows and columns of B_i^{-1} are indexed by V_i and E'_i, respectively. We use the concept of *circulant matrix*, that is, a matrix where each row is rotated one element to the right relative to the preceding row.

Lemma 4. *Let $i \in I_0 \cup I_1$, $u \in V_i$ and $e \in E'_i$. If $u = z_0, z_1, \ldots, z_s = \kappa_u$ is the unique path in C_i between u and κ_u then*

$$B_i^{-1}(u, e) = \begin{cases} (-1)^s H^{-1}(\kappa_u, e), & \text{if } e \in \kappa(C_i), \\ (-1)^j, & \text{if } e = z_j z_{j+1}, j = 0, 1, \ldots, s-1, \\ 0, & \text{otherwise,} \end{cases}$$

where $H^{-1} = [1]$ (if $i \in I_0$) or H^{-1} is a circulant matrix with the first row having the form

$$\frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & 1 \end{bmatrix}. \tag{10}$$

Proof: First, we characterize matrix H^{-1}. For $i \in I_0$, Lemma 3 trivially yields $H^{-1} = H = [1]$. For $i \in I_1$, the same lemma states that H is the edge-vertex incidence matrix

of an odd cycle. Then, we can arrange its rows and columns to obtain a circulant matrix whose first row is $\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$, thus giving the form (10) of H^{-1}.

Now, we prove the entries of B_i^{-1} by noting that $B_i^{-1}(u,e)$ is an entry of the $(s+1)$-th row of the block matrix (8). We use induction on $s = 0, 1, \ldots, p$, where p is the maximum distance between a vertex of C_i and its kernel. If $s = 0$, the expression of $B_i^{-1}(u,e)$ trivially follows. For $s > 0$, equations (9) imply that the non-zero part of the $(s+1)$-th row of (8) is

$$\begin{aligned} & \begin{bmatrix} J_{s1} & J_{s2} & \cdots & J_{s,s-1} & J_{ss} & I_{n_s} \end{bmatrix} \\ & = \; -X_s \begin{bmatrix} J_{s-1,1} & J_{s-1,2} & \cdots & J_{s-1,s-1} & I_{n_{s-1}} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & I_{n_s} \end{bmatrix}. \end{aligned}$$

Then, the form of X_s given by Lemma 3 leads to

$$B_i^{-1}(u,e) = \begin{cases} -B_i^{-1}(z_1,e), & \text{if } e \neq uz_1, \\ 1, & \text{if } e = uz_1. \end{cases} \tag{11}$$

Using the induction hypothesis, we get

$$B_i^{-1}(z_1,e) = \begin{cases} (-1)^{s-1} H^{-1}(\kappa_u, e), & \text{if } e \in \kappa(C_i), \\ (-1)^{j-1}, & \text{if } e = z_j z_{j+1}, j = 1, \ldots, s-1, \\ 0, & \text{otherwise.} \end{cases}$$

The expression of $B_i^{-1}(u,e)$ then follows from (11). □

Corollary 1. *For $i \in I_0$, B_i^{-1} is a $\{0, \pm 1\}$-matrix. For $i \in I_1$, an entry of B_i^{-1} is equal to $\pm 1/2$, if it is in a column indexed by the edges of $\kappa(C_i)$, or belongs to $\{0, \pm 1\}$, otherwise.*

Regarding the sum of the entries in a row of B_i^{-1}, we have that

Lemma 5. *For $i \in I_0$, $B_i^{-1}\mathbf{1} \in \{0,1\}^{|V_i|}$. For $i \in I_1$, $B_i^{-1}\mathbf{1} = (1/2)\mathbf{1}$.*

Proof: By (10), we trivially have

$$H^{-1}\mathbf{1} = s_i \mathbf{1},$$

where $s_i = 1$ (if $i \in I_0$) or $s_i = 1/2$ (if $i \in I_1$). With respect to a row of each submatrix J_{lj}, for $l = 1, 2, .., p$ and $j = 1, 2, \ldots, l$, it is clearly $(-1)^{l-j+1}$ times either a row of H^{-1} (if $j = 1$) or a row of $I_{n_{j-1}}$ (if $j > 1$). This is because any matrix X_t appearing in (9) has exactly one nonzero entry (equal to 1) in each row, according to Lemma 3. Therefore, for $l = 1, 2, \ldots, p$, it follows that

$$I_{n_l}\mathbf{1} + \sum_{j=1,\ldots,l} J_{lj}\mathbf{1} = (1 + (-1)^l s_i)\mathbf{1} + (-1)^{l+1} \sum_{j=2,\ldots,l} (-1)^j \mathbf{1} = \begin{cases} (1 - s_i)\mathbf{1}, & \text{if } l \text{ is odd,} \\ s_i \mathbf{1}, & \text{if } l \text{ is even.} \end{cases}$$

The two expressions above lead to the claimed result. □

4 The Corner polyhedron

The corner polyhedron associated with a basis of the stable set problem is the convex hull of the points satisfying the relaxation of (2) where the non-negativity constraints on the basic variables are discarded. Since each variable in y_B appears in only one equation and no other constraint, we can drop these variables and the corresponding equalities. If $V_B = \emptyset$, then we trivially have

$$\text{corner}(B) = \{(x, y) : Ax + y = \mathbf{1}, x \geq 0\} = \mathbb{R}^n_+ \times \mathbb{R}^m. \tag{12}$$

Otherwise, in order to express the remaining constraints in (2), let us partition the non-basic matrix N. According to the organization of B given by (3)–(5), the submatrix N can be partitioned as

$$N = \begin{bmatrix} \bar{N} & I \\ Q & 0 \end{bmatrix}, \tag{13}$$

where the first group of columns is indexed by the vertices in V_N and the second one is indexed by the edges in E_N.

Using (6) and (13), the corner polyhedron corner(B) is given by the convex hull of the set

$$P = \{(x, y) : x_B = \bar{B}^{-1}\mathbf{1} - \bar{B}^{-1}\bar{N}x_N - \bar{B}^{-1}y_N,\ x_N \geq 0, y_N \geq 0, x \in \mathbb{Z}^n\}, \tag{14}$$

We can further simplify the expression of P as follows. Let us partition $\bar{N}$ according to the structure of $\bar{B}$ shown in (4) to get

$$\bar{N} = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_k \end{bmatrix}.$$

Also, denote by x_{B_i} and y_{N_i} the vectors of variables indexed by the vertices and edges of C_i, respectively, for all $i = 1, 2, \ldots, k$. Notice that, for $i \in I_1$, we have $N_i = 0$ because every row of B_i contains two entries equal to 1. Then, we can rewrite

$$P = P' \times \prod_{i=1,\ldots,k'} P_i,$$

where

$$\begin{aligned} P' &= \{((x_{B_i}, y_{N_i})_{i \in I_0}, x_N, y_B) : x_{B_i} = B_i^{-1}\mathbf{1} - B_i^{-1}N_i x_N - B_i^{-1}y_{N_i}, \\ &\qquad x_N \in \mathbb{Z}^{|V_N|}_+,\ x_{B_i} \in \mathbb{Z}^{|V_i|}, y_{N_i} \geq 0, i \in I_0\}, & (15) \\ P_i &= \{(x_{B_i}, y_{N_i}) : x_{B_i} = B_i^{-1}\mathbf{1} - B_i^{-1}y_{N_i},\ y_{N_i} \geq 0, x_{B_i} \in \mathbb{Z}^{|V_i|}\}, \quad i \in I_1. & (16) \end{aligned}$$

The convex hull of P' readily follows from the integrality of the matrices involved in its definition.

Lemma 6. *The convex hull of P' is its linear relaxation.*

Proof: Since the integrality of x implies the integrality of y, we can assume y_{N_i} integer in (15). In addition, using Corollary 1 for $i \in I_0$ and the fact N_i is a $\{0,1\}$-matrix, we conclude that matrices B_i^{-1}, $B^{-1}\mathbf{1}$ and $B_i^{-1}N_i$ are integral. Therefore, the result follows. $\square$

Now, let $i \in I_1$. In order to describe $\text{conv}(P_i)$, let us partition $-B_i^{-1} = [R \quad T]$, where the submatrix R is indexed by the edges of $\kappa(C_i)$. By Corollary 1, all entries of R belong to $\{\pm 1/2\}$ whereas all entries of T are in $\{0, \pm 1\}$. Moreover, we can assume that the variables in y_{N_i} related to T are integral. Since $B_i^{-1}\mathbf{1} = (1/2)\mathbf{1}$ by Lemma 5, these facts lead to the following set equivalent to P_i:

$$\bar{P}_i = \left\{ (\bar{x}, \bar{y}) : \bar{x} = (1/2)\mathbf{1} + \sum_{e \in \kappa(C_i)} r^e y_e,\ \bar{x} \in \mathbb{Z}^{|V_i|},\ \bar{y} = (y_e)_{e \in \kappa(C_i)} \geq 0 \right\}, \tag{17}$$

where $r^e \in \{\pm 1/2\}^{|V_i|}$.

Toward our end, we first analyze the minimal valid inequalities of $\text{conv}(\bar{P}_i)$. A non-trivial valid inequality of $\text{conv}(\bar{P}_i)$ has the form $\sum_{e \in \kappa(C_i)} \alpha_e y_e \geq 1$ with $\alpha_e \geq 0$, and it is minimal if there is no valid inequality $\sum_{e \in \kappa(C_i)} \alpha'_e y_e \geq 1$ with $\alpha' \leq \alpha$ and $\alpha'_e < \alpha_e$, for some $e \in \kappa(C_i)$. The minimal valid inequalities of $\text{conv}(\bar{P}_i)$ include all the facet-defining inequalities necessary for its description and, consequently, for the description of $\text{conv}(P_i)$.

Lemma 7. *Let $i \in I_1$. A minimal valid inequality for conv($\bar{P}_i$) has the form $\sum_{e \in \kappa(C_i)} \alpha_e y_e \geq 1$, where $\alpha_e \geq 1$ for all $e \in \kappa(C_i)$.*

Proof: Any minimal valid inequality for $\text{conv}(\bar{P}_i)$ corresponds to a maximal lattice-free convex set C [7]. Besides, as we can assume that C contains $f = (1/2)\mathbf{1}$ as an interior point [15], the same theorem in [7] establishes that each coefficient α_e of the inequality is such that $f + (1/\alpha_e)r_e$ is a point in the boundary of C. Since $f + r_e$ is an integral point, for all $e \in \kappa(C_i)$, and C is a lattice free convex set, it follows that $1/\alpha_e \leq 1$. $\square$

Actually, we can show that the non-zero coefficients in a minimal valid inequality of $\text{conv}(P_i)$ are equal to 1 and can be obtained by the Chvátal procedure, as follows.

Lemma 8. *Let $i \in I_1$. The inequality $\sum_{e \in \kappa(C_i)} y_e \geq 1$ is valid for conv(P_i) and is a $\{0, 1/2\}$-Chvátal inequality for $S(G)$.*

Proof: Consider the equalities

$$B_i x_{B_i} + y_{N_i} = \mathbf{1} \tag{18}$$

that appears in (16) and take a linear combination with $\lambda \in \{0, 1/2\}^{|E_i|}$ and $\lambda_e = 1/2$ if and only if $e \in \kappa(C_i)$. Since $y_{N_i} \geq 0$ and $x_{B_i} \in \mathbb{Z}^{|V_i|}$ are constraints of P_i, the $\{0, 1/2\}$-Chvátal inequality $\lfloor \lambda B_i \rfloor x_{B_i} \leq \lfloor \lambda \mathbf{1} \rfloor$ or, equivalently, $\sum_{u \in \kappa(C_i)} x_u \leq (n_0 - 1)/2$, is valid for $\text{conv}(P_i)$, where n_0 is the (odd) number of vertices of $\kappa(C_i)$. Then, using equalities $y_e = 1 - x_u - x_v$, for all $e = uv \in \kappa(C_i)$, included in (18), we get the desired inequality in the y variables. Moreover, since (18) is a subset of the equations in (1), the second part of the statement follows. $\square$

Lemma 9. *Let $i \in I_1$. Then, conv$(P_i) = \{(x_{B_i}, y_{N_i}) : x_{B_i} = B_i^{-1}\mathbf{1} - B_i^{-1} y_{N_i}, \sum_{e \in \kappa(C_i)} y_e \geq 1,\ y_{N_i} \geq 0\}$.*

Proof: Consider a description of conv(P_i) given by minimal valid inequalities. By the relationship between P_i and $\bar{P}_i$ and Lemma 7, any inequality in this description has the form $\sum_{e \in \kappa(C_i)} \alpha_e y_e \geq 1$, where $\alpha_e \geq 1$ for all $e \in \kappa(C_i)$. By Lemma 8, we conclude that $\alpha_e = 1$, for all $e \in \kappa(C_i)$, and the result follows. □

Lemmas 6 and 9 and the trivial case (12) give the following description of the corner polyhedron.

Theorem 2. *For every $B \in \mathcal{B}$, the corner polyhedron of (2) associated to B is*

$$\text{corner}(B) = \left\{ (x, y) : Ax + y = \mathbf{1}, x_N \geq 0, y_N \geq 0, \sum_{e \in \kappa(C_i)} y_e \geq 1, i \in I_1 \right\}.$$

Theorem 2 implies that each non-trivial inequality necessary to describe the corner polyhedron is a fractional Gomory cut that can be derived from a row of the simplex tableau related to a basic vertex in the cycle of a 1-tree component.

Even if we consider a tighter relaxation of (2), where only constraints $y_B \geq 0$ are discarded, we cannot get stronger cuts. Indeed, if we keep the non-negativity of the (basic) variables associated with the vertices of the 1-tree component C_i ($i \in I_1$), the set conv(P_i) turns out to be the stable set polytope of C_i, whose description is the same as that one presented in Lemma 9 together with $x_{B_i} \geq 0$. Similarly, maintaining the non-negativity constraints on the (basic) variables related to vertices of the tree components leads to a redefinition of P' by the inclusion of $x_{B_i} \geq 0$, for all $i \in I_0$. Again, the convex hull of this restricted set is its linear relaxation. Therefore, only the trivial inequalities $x_B \geq 0$ can be obtained in addition to those describing corner(B).

5 The Chvátal closure

Lemma 8 and Theorem 2 show that the only non-trivial inequalities needed to define corner(B) are $\{0, 1/2\}$-Chvátal inequalities. They all have the form $\sum_{e \in C} y_e \geq 1$, where C is an odd cycle G. If this cycle is not an induced subgraph of G, this inequality is dominated by $\sum_{e \in C'} y_e \geq 1$, where C' is an odd cycle of G induced by a subset of vertices of C. This is the key point to characterizing the Chvátal closure of $S(G)$, as follows.

Let $\mathcal{C}$ denote the set of all the induced odd cycles of G. For every $C \in \mathcal{C}$, form the submatrix B_C of $[A \quad I]$ given by

$$B_C = \begin{bmatrix} A_C & 0 \\ 0 & I \end{bmatrix},$$

where A_C is the edge-vertex incidence matrix of C and I is related to the edges of G not in C. Notice that B_C is invertible and $B_C^{-1}\mathbf{1} \in \{1/2, 1\}^m$, which implies that B_C is a feasible basis. Let us denote by $\mathcal{B}_+ = \{B \in \mathcal{B} : B^{-1}\mathbf{1} \geq 0\}$ the set of feasible basis and

by $\mathcal{B}_\mathcal{C} = \{B_C : C \in \mathcal{C}\}$ its subset corresponding to $\mathcal{C}$. Also, for any $\mathcal{B}' \subseteq \mathcal{B}$, let corner($\mathcal{B}'$) stand for the intersection of the corner polyhedra associated to all bases in $\mathcal{B}'$, that is,

$$\text{corner}(\mathcal{B}') = \bigcap_{B \in \mathcal{B}'} \text{corner}(B).$$

Lemma 10. *corner*($\mathcal{B}$) $\subseteq$ *corner*($\mathcal{B}_+$) $\subseteq \mathbb{R}^n_+ \times \mathbb{R}^m_+$.

Proof: We only need to prove the second inclusion. First, note that corner($\mathcal{B}_+$) $\subseteq$ corner(I) $\subseteq \mathbb{R}^n_+ \times \mathbb{R}^m$. Now, for each edge $e = uv$ of G, consider the submatrix B_e of $[A \quad I]$ defined by the column of A indexed by u and the columns of I indexed by $E \setminus \{e\}$. Notice that $B_e \in \mathcal{B}_+$. In addition, by Theorem 2, $y_e \geq 0$ is a valid inequality for corner(B_e). Since corner($\mathcal{B}_+$) $\subseteq \cap_{e \in E}$corner(B_e), the result follows. □

Lemma 11. *corner*($\mathcal{B}$) $=$ *corner*($\mathcal{B}_+$) $= \bar{S}(G)$, *where*

$$\bar{S}(G) = \left\{ (x,y) \geq 0 : Ax + y = \mathbf{1},\ \sum_{e \in C} y_e \geq 1, C \in \mathcal{C} \right\}.$$

Proof: If $\mathcal{C} = \emptyset$, the result trivially follows by Theorem 2 and Lemma 10. The same statements imply that

$$\text{corner}(\mathcal{B}) \subseteq \text{corner}(\mathcal{B}_+) \subseteq \text{corner}(\mathcal{B}_\mathcal{C}) \cap (\mathbb{R}^n_+ \times \mathbb{R}^m_+) = \bar{S}(G).$$

To obtain equalities above, it suffices to note that there is always a subset of the vertices of any non-induced odd cycle C that defines an induced odd cycle $C \in \mathcal{C}$. Then, the inequality $\sum_{e \in C'} y_e \geq 1$ is tighter than $\sum_{e \in C} y_e \geq 1$, which shows that $\bar{S}(G) \subseteq$ corner($\mathcal{B}$). □

Lemma 11 and the fact that all inequalities needed in the description of corner($\mathcal{B}$) are $\{0, 1/2\}$-Chvátal inequalities lead to the following more complete version of the theorem stated in the introduction.

Theorem 3. *For the stable set formulation* (1)*, $\bar{S}(G)$, the $\{0, 1/2\}$-Chvátal closure, the Chvátal closure, the split closure, corner*($\mathcal{B}_+$) *and corner*($\mathcal{B}$) *are identical.*

Finally, it is worth relating the above result to the class of h-perfect graphs, which includes the series-parallel graphs [6, 13]. A graph G is h-perfect if its stable set polytope is equal to $\bar{S}(G)$ (together with $x_v \leq 1$, for every vertex v that defines a trivial connected component). By Theorem 3, we get the following characterization.

Corollary 2. *A graph G is h-perfect if, and only if, the stable set polytope of G is equal to its Chvátal closure.*

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